

# SOME TYPE I SOLUTIONS OF RICCI FLOW WITH ROTATIONAL SYMMETRY

JIAN SONG

**ABSTRACT.** We prove that the Ricci flow on  $\mathbb{CP}^n$  blown-up at one point starting with any rotationally symmetric Kähler metric must develop Type I singularities. In particular, if the total volume does not go to zero at the singular time, the parabolic blow-up limit of the Type I Ricci flow along the exceptional divisor is a complete non-flat shrinking gradient Kähler-Ricci soliton on a complete Kähler manifold homeomorphic to  $\mathbb{C}^n$  blown-up at one point.

## 1. Introduction

In this paper, we study the Ricci flow on Kähler manifolds defined by

$$X_{n,k} = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-k))$$

for  $k, n \in \mathbb{N}^+$ . Such manifolds are holomorphic  $\mathbb{CP}^1$  bundle over the projective space  $\mathbb{CP}^{n-1}$ . They are called Hirzebruch surfaces when  $n = 2$  and  $X_{n,1}$  is exactly  $\mathbb{CP}^n$  blown-up at one point. The maximal compact subgroup of the automorphism group of  $X_{n,k}$  is given by  $G_{n,k} = U(n)/\mathbb{Z}_k$  ([2]).

The unnormalized Ricci flow introduced by Hamilton [9] is defined on a Riemannian manifold  $M$  starting with a Riemannian metric  $g_0$  by

$$(1.1) \quad \frac{\partial g}{\partial t} = -Ric(g), \quad g(0) = g_0.$$

We apply the Ricci flow (1.1) to  $X_{n,k}$  with a  $G_{n,k}$ -invariant initial Kähler metric. In [18], it is shown that the Ricci flow (1.1) must develop finite time singularity and it either shrinks to a point, collapses to  $\mathbb{CP}^{n-1}$  or contracts an exceptional divisor, in Gromov-Hausdorff topology.

When the flow shrinks to a point,  $X_{n,k}$  is a Fano manifold and  $1 \leq k < n$ . It is shown by Zhu [28] that the flow must develop Type I singularities and the rescaled Ricci flow converges in Cheeger-Gromov-Hamilton sense to the unique compact Kähler-Ricci soliton on  $X_{n,k}$  constructed in [8, 3, 24].

When the flow collapses to  $\mathbb{CP}^{n-1}$ , it is shown by Fong [7] that the flow must develop Type I singularities and the rescaled Ricci flow converges in Cheeger-Gromov-Hamilton sense to the ancient solution that splits isometrically as  $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ .

Our main result is to show that the flow must also develop Type I singularities when it does not collapse and the blow-up limit is a nontrivial complete shrinking

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gradient Kähler-Ricci soliton. Combined with the results of Zhu [28] and Fong [7], we have the following theorem.

**Theorem 1.1.** *Let  $X$  be  $\mathbb{CP}^n$  blown-up at one point. Then the Ricci flow on  $X$  must develop Type I singularities for any  $U(n)$ -invariant initial Kähler metric.*

Let  $g(t)$  be the smooth solution defined on  $t \in [0, T)$ , where  $T \in (0, \infty)$  is the singular time. For every  $K_j \rightarrow \infty$ , we consider the rescaled Ricci flow  $(X, g_j(t'))$  defined on  $[-K_j T, 0)$  by

$$g_j(t') = K_j g(T + K_j^{-1} t').$$

Then one and only one of the following must occur.

- (1) If  $\liminf_{t \rightarrow T} (T - t)^{-1} \text{Vol}(g(t)) = \infty$ , then  $(X, g_j(t'), p)$  subconverges in Cheeger-Gromov-Hamilton sense to a complete shrinking non-flat gradient Kähler-Ricci soliton on a complete Kähler manifold homeomorphic to  $\mathbb{C}^n$  blown-up at one point, for any  $p$  in the exceptional divisor.
- (2) If  $\liminf_{t \rightarrow T} (T - t)^{-1} \text{Vol}(g(t)) \in (0, \infty)$ , then  $(X, g_j(t'), p_j)$  subconverges in Cheeger-Gromov-Hamilton sense to  $(\mathbb{C}^{n-1} \times \mathbb{CP}^1, g_{\mathbb{C}^{n-1}} \oplus (-t') g_{FS})$ , where  $g_{\mathbb{C}^{n-1}}$  is the standard flat metric on  $\mathbb{C}^{n-1}$  and  $g_{FS}$  the Fubini-Study metric on  $\mathbb{CP}^1$  for any sequence of points  $p_j$  [7].
- (3) If  $\liminf_{t \rightarrow T} (T - t)^{-1} \text{Vol}(g(t)) = 0$ , then  $(X, g_j(t'))$  converges in Cheeger-Gromov-Hamilton sense to the unique compact shrinking Kähler-Ricci soliton on  $\mathbb{CP}^n$  blown-up at one point [28].

The generalization of Theorem 1.1 for  $X_{n,k}$  is given in section 6. In order to exclude Type II singularities, we first prove a lower bound for the holomorphic bisectional curvature and then we apply Cao's splitting theorem for the Kähler Ricci flow with nonnegative holomorphic bisectional curvature [4]. Theorem 1.1 gives evidence that the Kähler-Ricci flow can only develop Type I singularities for Kähler surfaces and if the flow does not collapse in finite time. Combined with the results of [18, 19], Theorem 1.1 verifies that the flow indeed performs a geometric canonical surgery with minimal singularities in the Kähler case. We also remark that the shrinking soliton as the pointed blow-up limit is trivial if the parabolic rescaling takes place at a fixed base point outside the exceptional divisor  $D_0$ . We believe that the blow-up limit should be the unique homothetically rotationally symmetric complete shrinking soliton on  $\mathbb{C}^2$  blown-up at one point constructed by Feldman-Ilmanen-Knopf in [6]. Unfortunately, we are unable to show that that limiting complete Kähler manifold is biholomorphic to  $\mathbb{C}^n$  blown-up at one point, although it has the same topological structure with the unitary group  $U(n)$  lying in the isometry group of the limiting soliton.

The organization of the paper is as follows. In section 2, we introduce the Calabi ansatz. In section 3, we obtain a lower bound for the holomorphic bisectional curvature. In section 4, we prove the flow must develop Type I singularities if non-collapsing. In section 5, we construct the blow-up limit. In section 6, we discuss some generalizations of Theorem 1.1.

We would also like to mention that we have been informed by Davi Maximo that he has a different approach to understand the singularity formation in similar settings [13].

## 2. Calabi symmetry

In this section, we introduce the Calabi ansatz on  $\mathbb{CP}^n$  blown-up at one point introduced by Calabi [2] (also see [3, 6, 18]). From now on, we let  $X$  be  $\mathbb{CP}^n$  blown-up at one point and it is in fact a  $\mathbb{CP}^1$  bundle over  $\mathbb{CP}^{n-1}$  given by

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)).$$

Let  $D_0$  be the exceptional divisor of  $X$  defined by the image of the section  $(1, 0)$  of  $\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$  and  $D_\infty$  be the divisor of  $X$  defined by the image of the section  $(0, 1)$  of  $\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ . Both the 0-section  $D_0$  and the  $\infty$ -section are complex hypersurfaces in  $X$  isomorphic to  $\mathbb{CP}^{n-1}$ . The Kähler cone on  $X$  is given by

$$\mathcal{K} = \{-a[D_0] + b[D_\infty] \mid 0 < a < b\}.$$

In particular, when  $n = 2$ ,  $D_0$  is a holomorphic  $S^2$  with self-intersection number  $-1$ .

Let  $z = (z_1, \dots, z_n)$  be the standard holomorphic coordinates on  $\mathbb{C}^n$ . Let  $\rho = \log |z|^2 = \log(|z_1|^2 + |z_2|^2 + \dots + |z_n|^2)$ . We consider a smooth convex function  $u = u(\rho)$  for  $\rho \in (-\infty, \infty)$  satisfying the following conditions.

- (1)  $u'' > 0$  for  $\rho \in (-\infty, \infty)$ .
- (2) There exist  $0 < a < b$  and smooth function  $u_0, u_\infty : [0, \infty) \rightarrow \mathbb{R}$  such that

$$u'_0(0) > 0, \quad u'_\infty(0) > 0,$$

$$u_0(e^\rho) = u(\rho) - a\rho, \quad u_\infty(e^{-\rho}) = u(\rho) - b\rho.$$

For any  $u$  satisfying the above conditions,  $\omega = \sqrt{-1}\partial\bar{\partial}u$  defines a smooth Kähler metric on  $\mathbb{C}^n \setminus \{0\}$  and it extends to a smooth global Kähler metric on  $\mathbb{CP}^n$  blown-up at one point in the Kähler class  $-a[D_0] + b[D_\infty]$ .

On  $\mathbb{C}^n \setminus \{0\}$ , the Kähler metric  $g$  induced by  $u$  is given by

$$(2.2) \quad g_{i\bar{j}} = e^{-\rho} u' \delta_{i\bar{j}} + e^{-2\rho} \bar{z}_i z_j (u'' - u').$$

Obviously, the Kähler metric  $g$  induced by  $u$  is invariant under the standard unitary  $U(n)$  transformations on  $\mathbb{C}^n$ .

We define the Ricci potential of  $\omega = \sqrt{-1}\partial\bar{\partial}u$  by

$$(2.3) \quad v = -\log \det g = n\rho - (n-1) \log u'(\rho) - \log u''(\rho).$$

and the Ricci tensor of  $g$  is given by

$$R_{i\bar{j}} = e^{-\rho} v' \delta_{i\bar{j}} + e^{-2\rho} \bar{z}_i z_j (v'' - v').$$

After applying a unitary transformation, we can assume  $z = (z_1, 0, \dots, 0)$  and then

$$\{g_{i\bar{j}}\} = e^{-\rho} \text{diag}\{u'', u', \dots, u'\}$$

$$R_{i\bar{j}} = \sqrt{-1} e^{-\rho} \text{diag}\{v'', v', \dots, v'\}.$$

The Calabi symmetry is preserved by the Ricci flow, in other words, the evolving Kähler metric is invariant under the  $U(n)$ -action if the Ricci flow starts with a  $U(n)$ -invariant Kähler metric on  $X$ .

In [18], it is shown that the Kähler-Ricci flow on  $X$  can be reduced to the following parabolic equation for  $u = u(\rho, t)$  for  $\rho \in \mathbf{R}$ .

$$(2.4) \quad \frac{\partial}{\partial t} u(\rho, t) = \log u''(\rho, t) + (n-1) \log u'(\rho, t) - n\rho + c_t,$$

where

$$c_t = -\log u''(0, t) - (n-1)u'(0, t)$$

and  $u'(\rho, t) = \frac{\partial}{\partial \rho} u(\rho, t)$ . The evolving Kähler form  $\omega(t)$  is then given by

$$\omega(t) = \sqrt{-1} \partial \bar{\partial} u(\rho, t).$$

It is also shown in [18] that if the initial Kähler class is given by  $-a_0[D_0] + b_0[D_\infty]$ , the evolving Kähler class is given by

$$[\omega(t)] = -a_t[D_0] + b_t[D_\infty], \quad a_t = a_0 - (n-1)t, \quad b_t = b_0 - (n+1)t.$$

In particular, we have an immediate bound for  $u'(\rho, t)$

$$(2.5) \quad \lim_{\rho \rightarrow -\infty} u'(\rho, t) = a_t, \quad \lim_{\rho \rightarrow \infty} u'(\rho, t) = b_t.$$

### 3. A lower bound for the holomorphic bisectional curvature

In this section, we will obtain a lower bound for the holomorphic bisectional curvature. We consider the Ricci flow (1.1) on  $X$  with a  $U(n)$ -invariant initial Kähler metric in the Kähler class  $-a_0[D_0] + b_0[D_\infty]$ . For our purpose, it suffices to consider the case

$$0 < a_0(n+1) < b_0(n-1).$$

This assumption is shown in [18] to be equivalent to the condition

$$\liminf_{t \rightarrow T} \text{Vol}(g(t)) > 0, \quad \text{or,} \quad \liminf_{t \rightarrow T} (T-t)^{-1} \text{Vol}(g(t)) = \infty$$

and then the Kähler-Ricci flow will contract the exceptional divisor  $D_0$  at the singular time

$$T = \frac{a_0}{n-1}.$$

We will assume through out this section that the initial Kähler class lies in  $-a_0[D_0] + b_0[D_\infty]$  with  $0 < a_0(n+1) < b_0(n-1)$ .

The following theorem is proved in [22].

**Theorem 3.1.** *For any relatively compact set  $K$  of  $X \setminus D_0$  and  $k > 0$ , there exists  $C_{K,k} > 0$  such that for all  $t \in [0, T)$ ,*

$$\|g(t)\|_{C^k(K, g_0)} \leq C_{K,k}.$$

It immediately implies that the Ricci flow converges in local  $C^\infty$  topology outside the exceptional divisor  $D_0$  as  $t \rightarrow T$ .

The evolution equations for  $u', u'', u'''$  are derived in [18] as below.

$$(3.6) \quad \frac{\partial}{\partial t} u' = \frac{u'''}{u''} + \frac{(n-1)u''}{u'} - n$$

$$(3.7) \quad \frac{\partial}{\partial t} u'' = \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + \frac{(n-1)u'''}{u'} - \frac{(n-1)(u'')^2}{(u')^2}$$

$$(3.8) \quad \begin{aligned} \frac{\partial}{\partial t} u''' &= \frac{u^{(5)}}{u''} - \frac{3u'''u^{(4)}}{(u'')^2} + \frac{2(u''')^3}{(u'')^3} + \frac{(n-1)u^{(4)}}{u'} \\ &\quad - \frac{3(n-1)u''u'''}{(u')^2} + \frac{2(n-1)(u'')^3}{(u')^3}. \end{aligned}$$

The following lemma is proved in [18] for the collapsing case when  $a_0(n+1) > b_0(n-1)$  and the same proof can be applied here. We include the proof for the sake of completeness.

**Lemma 3.1.** *There exists  $C > 0$  such that for all  $t \in [0, T)$  and  $\rho \in (-\infty, \infty)$ ,*

$$(3.9) \quad (n-1)(T-t) \leq u' \leq C$$

and

$$(3.10) \quad 0 \leq \frac{u''}{u'} \leq C, \quad -C \leq \frac{u'''}{u''} \leq C.$$

*Proof.* The estimate (3.9) follows from the monotonicity of  $u'$  with  $a_t < u' \leq b_t$  and  $a_t = (n-1)(T-t)$ .

We apply the maximum principle to prove (3.10). It is straightforward to verify that for all  $t \in [0, T)$ ,

$$\begin{aligned} \lim_{\rho \rightarrow -\infty} \frac{u''(\rho, t)}{u'(\rho, t)} &= \lim_{\rho \rightarrow \infty} \frac{u''(\rho, t)}{u'(\rho, t)} = 0 \\ \lim_{\rho \rightarrow -\infty} \frac{u'''(\rho, t)}{u''(\rho, t)} &= 1, \quad \lim_{\rho \rightarrow \infty} \frac{u'''(\rho, t)}{u''(\rho, t)} = -1. \end{aligned}$$

Let  $H = \frac{u''}{u'}$ .  $H$  is strictly positive for all  $\rho \in (-\infty, \infty)$  and  $t \in [0, T)$ . The evolution for  $H$  is given by

$$\frac{\partial H}{\partial t} = \frac{H''}{u''} + \frac{2H'}{u'} - \frac{2H^2 - H}{u'}.$$

Therefore  $\sup_{\rho \in (-\infty, \infty), t \in [0, T)} H \leq C$  for some uniform constant  $C > 0$  by applying the maximum principle.

Let  $G = \frac{u'''}{u''}$ . Then the evolution for  $G$  is given by

$$\frac{\partial G}{\partial t} = \frac{1}{u''} G'' + \left( \frac{n-1}{u'} - \frac{u'''}{(u'')^2} \right) G' - \frac{2(n-1)u''}{(u')^2} \left( G - \frac{u''}{u'} \right).$$

Therefore  $\sup_{\rho \in (-\infty, \infty), t \in [0, T)} |G| \leq C$  for some uniform constant  $C > 0$  by combining the maximum principle and the uniform upper bound for  $H$ .

□

By taking the trace, we obtain an explicit expression for the scalar curvature

$$(3.11) \quad R = -\frac{\frac{\partial u''}{\partial t}}{u''} - \frac{(n-1)\frac{\partial u'}{\partial t}}{u'} = -\frac{u^{(4)}}{(u'')^2} + \frac{(u''')^2}{(u'')^3} - \frac{2(n-1)u'''}{u'u''} - \frac{(n-1)(n-2)u''}{(u')^2} + \frac{n(n-1)}{u'}.$$

**Corollary 3.1.** *There exists  $C > 0$  such that for all  $\rho \in (-\infty, \infty)$  and  $t \in [0, T)$ ,*

$$(3.12) \quad -\frac{u^{(4)}}{(u'')^2} + \frac{(u''')^2}{(u'')^3} \geq -\frac{C}{T-t}.$$

*Proof.* Since the scalar curvature  $R$  is uniformly bounded below, there exists  $C_1 > 0$  such for all  $t \in [0, T)$  and  $\rho \in (-\infty, \infty)$ ,

$$-\frac{u^{(4)}}{(u'')^2} + \frac{(u''')^2}{(u'')^3} - \frac{2(n-1)u'''}{u'u''} - \frac{(n-1)(n-2)u''}{(u')^2} + \frac{n(n-1)}{u'} \geq -C_1.$$

There also exist  $C_2, C_3 > 0$  such that

$$u' \geq C_2(T-t)$$

and

$$\left| \frac{u''}{u'} \right| + \left| \frac{u'''}{u''} \right| \leq C_3.$$

The estimate (3.12) immediately follows from the above estimates.  $\square$

The holomorphic bisectional curvature  $R_{i\bar{j}k\bar{l}}$  is computed in [3] and is given by

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= e^{-2\rho}(u' - u'')(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}) \\ &\quad + e^{-2\rho}(3u'' - 2u' - u''')(\delta_{ij}\delta_{kl1} + \delta_{il}\delta_{kj1} + \delta_{kl}\delta_{ij1} + \delta_{kj}\delta_{il1}) \\ &\quad + e^{-2\rho}\left(6u''' - 11u'' - u^{(4)} + 6u' + \frac{(u'' - u''')^2}{u''}\right)\delta_{ijkl1} \\ &\quad + e^{-2\rho}\frac{(u' - u'')^2}{u'}(\delta_{ij\hat{1}}\delta_{kl1} + \delta_{il\hat{1}}\delta_{kj1} + \delta_{kl\hat{1}}\delta_{ij1} + \delta_{kj\hat{1}}\delta_{il1}) \end{aligned}$$

Here  $\delta_{ij1}$  and  $\delta_{ijkl1}$  vanish unless all the indices are 1, while  $\delta_{ij\hat{1}}$  vanishes unless  $i = j \neq 1$ .

For any point  $p$  on  $\mathbb{C}^n \setminus \{0\}$ , we can assume the coordinates at  $p$  are given by  $z(p) = (z_1, \dots, z_n) = (z_1, 0, \dots, 0)$  after a unitary transformation.

Then all the nonvanishing terms of the holomorphic bisectional curvature are given by

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= e^{-2\rho}\left(-u^{(4)} + \frac{(u''')^2}{u''}\right) \\ R_{k\bar{k}k\bar{k}} &= 2e^{-2\rho}(u' - u''), \quad k > 1 \\ R_{1\bar{1}k\bar{k}} &= e^{-2\rho}\left(-u''' + \frac{(u'')^2}{u'}\right), \quad k > 1 \\ R_{k\bar{k}l\bar{l}} &= e^{-2\rho}(u' - u''), \quad k > 1, l > 1, \quad k \neq l. \end{aligned}$$

**Lemma 3.2.** *There exists  $C > 0$  such that on for all  $t \in [0, T)$ ,  $p = (z_1, 0, \dots, 0)$  and  $i, j, k, l$ , we have at  $(p, t)$ ,*

$$(3.13) \quad R_{i\bar{j}k\bar{l}} \geq -\frac{C}{T-t}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}).$$

Furthermore,

$$(3.14) \quad |R_{i\bar{j}k\bar{l}}| \leq \frac{C}{T-t}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$$

unless  $i = j = k = l = 1$ .

*Proof.* Since  $p = (z_1, 0, \dots, 0)$ , it suffices to verify the estimates for  $R_{1\bar{1}1\bar{1}}$ ,  $R_{1\bar{1}k\bar{k}}$  and  $R_{k\bar{k}l\bar{l}}$  for  $k, l = 2, \dots, n$ .

Let  $Q_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$ . Then

$$\begin{aligned} Q_{1\bar{1}1\bar{1}} &= 2e^{-2\rho}(u'')^2 \\ Q_{k\bar{k}k\bar{k}} &= 2e^{-2\rho}(u')^2, \quad k > 1 \\ Q_{1\bar{1}k\bar{k}} &= e^{-2\rho}u'u'', \quad k > 1 \\ Q_{k\bar{k}l\bar{l}} &= e^{-2\rho}(u')^2, \quad k > 1, l > 1, \quad k \neq l. \end{aligned}$$

Comparing  $R_{i\bar{j}k\bar{l}}$  and  $Q_{i\bar{j}k\bar{l}}$ , the lemma follows immediately.  $\square$

**Proposition 3.1.** *The holomorphic bisectional curvature is uniformly bounded below by  $-C(T-t)^{-1}$  on  $X \times [0, T)$  for some fixed constant  $C > 0$ .*

*Proof.* It suffices to calculate the lower bound of the holomorphic bisectional curvature at a point  $p = (z_1, 0, \dots, 0)$  and  $t \in [0, T)$ . Let  $V = V^i \frac{\partial}{\partial z_i}$  and  $W = W^i \frac{\partial}{\partial z_i}$  be two vectors in  $TX_p$ . Then there exists  $C > 0$  such that

$$\begin{aligned} & R_{i\bar{j}k\bar{l}} V^i V^{\bar{j}} W^k W^{\bar{l}} \\ &= R_{1\bar{1}1\bar{1}} V^1 V^{\bar{1}} W^1 W^{\bar{1}} + (1 - \delta_{ijkl}) R_{i\bar{j}k\bar{l}} W^k W^{\bar{l}} \\ &\geq -\frac{2C}{T-t} g_{1\bar{1}} g_{1\bar{1}} |V^1|^2 |W^1|^2 - \frac{C}{T-t} (g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}) |V^i| |V^{\bar{j}}| |W^k| |W^{\bar{l}}| \\ &\geq -\frac{4C}{T-t} |V|_g^2 |W|_g^2. \end{aligned}$$

$\square$

**Definition 3.1.** *Let  $g$  be a Kähler metric on a Kähler manifold  $M$ . At each point  $p \in X$ , we can choose the normal coordinates at  $p$  such that for  $i, j = 1, \dots, n$ ,  $g_{i\bar{j}}(p) = \delta_{ij}$  is the identity matrix and*

$$R_{i\bar{j}}(p) = \delta_{ij} \lambda_j.$$

We define the  $k^{\text{th}}$  symmetric polynomial of Ricci curvature of  $g$  at  $p$  by

$$(3.15) \quad \sigma_k = \sigma_k(\text{Ric}(g)) = \sum_{j_1 < j_2 < \dots < j_k} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}$$

for  $1 \leq k \leq n$ .

The next proposition gives a uniform bound for  $\sigma_k$  in terms of the curvature tensor  $R_{1\bar{1}1\bar{1}}$  at each point  $z = (z_1, 0, \dots, 0)$ .

**Proposition 3.2.** *There exists  $C > 0$  such that for all  $(p, t) \in X \times [0, \infty)$ ,*

$$(3.16) \quad |\sigma_k(p, t)| \leq \frac{C|Rm(p, t)|}{(T-t)^{k-1}}.$$

*Proof.* For any point  $p \in \mathbb{C}^n \setminus \{0\}$ , we can assume that  $p = (z_1, 0, \dots, 0)$ . Then the eigenvalues of  $Ric(g)$  at  $p$  with respect to  $g$  are given by

$$\begin{aligned} \lambda_1 &= -\frac{\frac{\partial u''}{\partial t}}{u''} = -\frac{u^{(4)}}{(u'')^2} + \frac{(u''')^2}{(u'')^3} - \frac{(n-1)u'''}{u'u''} + \frac{(n-1)u''}{(u')^2} \\ \lambda_2 &= \dots = \lambda_n = -\frac{\frac{\partial u'}{\partial t}}{u'} = -\frac{u'''}{u'u''} - \frac{(n-1)u''}{(u')^2} + \frac{n}{u'}. \end{aligned}$$

Then  $(T-t)|\lambda_j|$  is uniformly bounded for  $j = 2, \dots, n$  and

$$|\sigma_k|(p, t) = \sum_{j_1 < j_2 < \dots < j_k} |\lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}| \leq C(T-t)^{-(k-1)} |\lambda_1| \leq C(T-t)^{-(k-1)} |Rm|_g(p, t).$$

□

**Lemma 3.3.** *For any  $p \in D_0$ , we have*

$$(3.17) \quad |Ric(p, t)|_{g(t)} \geq \frac{1}{T-t}.$$

*Proof.* It suffices to compute  $e^{-\rho}v'$  which is one of the eigenvalues in the Ricci tensors since  $D_0 = \{\rho = -\infty\}$ .

$$\begin{aligned} \lim_{\rho \rightarrow -\infty} e^{-\rho}v'(\rho) &= -\lim_{\rho \rightarrow -\infty} \frac{u'''}{u'u''} - \lim_{\rho \rightarrow -\infty} \frac{(n-1)u''}{(u')^2} + \lim_{\rho \rightarrow -\infty} \frac{n}{u'} \\ &= (n-1) \lim_{\rho \rightarrow -\infty} (u')^{-1} \\ &= \frac{1}{T-t}. \end{aligned}$$

Therefore  $|Ric|_g$  is uniformly bounded below by  $(T-t)^{-1}$  along the exceptional divisor  $D_0$ .

□

#### 4. Type I singularities

In this section, we prove that the Ricci flow must develop Type I singularities with the same assumptions in section 4.

Let's first recall the definition for a Type I singularity of the Ricci flow.

**Definition 4.1.** *Let  $(M, g(t))$  be a smooth solution of the Ricci flow (1.1) for  $t \in [0, T)$  with  $T < \infty$ . It is said to develop a Type I singularity at  $T$  if it cannot be smoothly extended past  $T$  and there exists  $C > 0$  such that for all  $t \in [0, T)$ ,*

$$(4.18) \quad \sup_M |Rm(g(t))|_{g(t)} \leq \frac{C}{T-t}.$$



The following splitting theorem is proved in [4] as a complex analogue of Hamilton's splitting theorem on Riemannian manifolds with nonnegative curvature operator [10].

**Theorem 4.1.** *Let  $g$  be a complete solution of the Kähler-Ricci flow on a noncompact simply connected Kähler manifold  $M$  of dimension  $n$  for  $t \in (-\infty, \infty)$  with bounded and nonnegative holomorphic bisectional curvature. Then either  $g$  is of positive Ricci curvature for all  $p \in M$  and all  $t \in (-\infty, \infty)$ , or  $(M, g)$  splits holomorphically isometrically into a product  $\mathbb{C}^k \times N^{n-k}$  ( $k \geq 1$ ) flat in  $\mathbb{C}^k$  direction and  $N$  being of nonnegative holomorphic bisectional curvature and positive Ricci curvature.*

We are now able to exclude Type II singularities.

**Theorem 4.2.** *Let  $X$  be  $\mathbb{CP}^n$  blown-up at one point and  $g(t)$  be the solution of the Kähler-Ricci flow on  $X$  starting with a  $U(n)$ -invariant Kähler metric  $g_0$ . If  $g_0$  lies in the Kähler class*

$$-a_0[D_0] + b_0[D_\infty]$$

*for  $0 < a_0(n+1) < b_0(n-1)$ . Then the flow develops Type I singularities at  $T = a_0/(n-1)$ .*

*Proof.* Suppose the flow develops Type II singularities. Let  $t_j$  be an increasing sequence converging to  $T = (n-1)a_0 > 0$  and  $p_j$  a sequence of points on  $X$  such that

$$K_j = |Rm(p_j, t_j)|_{g(t_j)} = \sup_X |Rm|_{g(t_j)}$$

and

$$\lim_{j \rightarrow \infty} (T - t_j)^{-1} K_j^{-1} = 0.$$

Applying the standard parabolic rescaling, we define

$$g_j(t) = K_j g(t_j + K_j^{-1}t).$$

After extracting a convergent subsequence,  $(X, g_j(t), p_j)$  converges in pointed Cheeger-Gromov-Hamilton sense to a complete eternal solution  $(X_\infty, g_\infty(t), p_\infty)$  on a complete Kähler manifold  $X_\infty$  of dimension  $n$ . Furthermore, by the lower bound of the holomorphic bisectional curvature of  $g(t)$  by Proposition 3.1, the limiting Kähler metric  $g_\infty(t)$  has nonnegative holomorphic bisectional curvature everywhere on  $X_\infty$ . On the other hand, the symmetric product of the Ricci curvature  $g_\infty$  vanishes everywhere in  $X_\infty$ ,

$$\sigma_k(Ric(g_\infty)) = 0$$

for  $2 \leq k \leq n$ . This implies that the Ricci curvature of  $g_\infty$  is not positive at each point of  $X_\infty$ . By applying the splitting theorem 4.1 for  $(n-1)$  times,  $(\tilde{X}_\infty, \tilde{g}_\infty, \tilde{p}_\infty)$ , the eternal solution on the universal cover of  $(X_\infty, g_\infty, p_\infty)$ , splits holomorphically isometrically into  $\mathbb{C}^{n-1} \times N$ , where  $N$  is a compact or complete Riemann surface with positive scalar curvature. By the classification of eternal solutions of real dimension 2 by Hamilton [11],  $(N, \tilde{g}_\infty(t)|_N)$  is a steady gradient soliton and hence it must be the cigar soliton. However, it violates Peralman's local non-collapsing [15], so does  $(X_\infty, g_\infty)$ . It then leads to a contradiction.  $\square$

### 5. Blow-up limits

In this section, we will prove that the blow-up limit of the Ricci flow near the singular time  $T$  along the exceptional divisor is a nontrivial complete shrinking gradient Kähler-Ricci soliton.

We first prove a diameter bound of the exceptional divisor  $D_0$ .

**Lemma 5.1.** *For all  $t \in [0, T)$ ,*

$$(5.19) \quad g(t)|_S = a_0(n-1)(T-t)g_{FS}.$$

and so

$$(5.20) \quad \text{diam}(S, g(t)|_{D_s}) = \alpha_n(a_0(n-1)(T-t))^{1/2}$$

where  $g_{FS}$  is a Fubini-Study metric on  $\mathbb{CP}^{n-1}$  and  $\alpha_n$  is the diameter of  $(\mathbb{CP}^{n-1}, g_{FS})$ .

*Proof.* The Kähler metric  $g(t)$  is the metric completion of the following metric on  $\mathbb{C}^n \setminus \{0\}$

$$\omega(t) = a_0(n-1)(T-t)\sqrt{-1}\partial\bar{\partial}\rho + \sqrt{-1}\partial\bar{\partial}u_0(e^\rho, t),$$

where  $u_0(\cdot, t)$  is smooth and for each  $t \in [0, T)$  with  $u'(0, t) > 0$ . Note that after extending  $\sqrt{-1}\partial\bar{\partial}\rho = \sqrt{-1}\partial\bar{\partial}\log|z|^2$  to  $\mathbb{CP}^n$  blown-up at one point, its restriction on  $D_0$  is exactly a Fubini-Study metric. The lemma then follows immediately.  $\square$

Now we can complete the proof of Theorem 1.1 by identifying the blow-up limit of the Ricci flow at the singular time.

**Proposition 5.1.** *Fix any  $p \in D_0$ . Then for every  $K_j \rightarrow \infty$ , the rescaled Ricci flows  $(X, g_j(t), p)$  defined on  $[-K_j T, 0)$  by*

$$g_j(t) = K_j g(T + K_j^{-1}t)$$

*subconverges in Cheeger-Gromov-Hamilton sense to a complete shrinking gradient Kähler-Ricci soliton on a complete Kähler manifold homeomorphic to  $\mathbb{C}^n$  blown-up at one point.*

*Proof.* We first show that the blow-up limit is a nontrivial complete shrinking soliton. Fix any point  $p \in D_0$  in the exceptional divisor. Since  $(X, g(t))$  is a Type I Ricci flow, the rescaled Ricci flow  $(X, g_t(t), p)$  always subconverges to a shrinking gradient soliton  $(X_\infty, g_\infty(t), p_\infty)$  in pointed Cheeger-Gromov-Hamilton sense, by the compactness result of Naber [14]. Such a limiting soliton cannot be flat because of Lemma 3.3. In particular,  $(X_\infty, g_\infty, p_\infty)$  is a complete shrinking gradient Kähler-Ricci soliton on a complete Kähler manifold  $X_\infty$ .

We now show that  $X_\infty$  is in fact homeomorphic to  $\mathbb{C}^n$  blown-up at one point. Fix a closed interval  $[a, b] \subset (-\infty, 0)$ , the rescaled Ricci flow  $g_j(t)$  restricted to  $D_0$  is uniformly equivalent to a fixed standard Fubini-Study metric on  $\mathbb{CP}^{n-1}$  for all  $j$  and  $t \in [a, b]$  by Lemma 5.1 and so there exist  $d, D > 0$  such that the diameter of  $D_0$  with respect to  $g_j(t)$  is uniformly bounded between  $d$  and  $D$  for all  $j$  and  $t \in [a, b]$ . We denote by

$$B_g(p, R)$$

the geodesic ball with respect to  $g$  centered at  $p$  with radius  $R$ . We then consider

$$\mathcal{B}_{j,t}(D_0, R) = \cup_{p \in D_0} B_{g_j(t)}(p, R)$$

for each  $t \in [a, b]$ . By choosing  $R$  sufficiently large, we have

$$B_{g_j(t)}(p, R) \subset \mathcal{B}_{j,t}(D_0, R) \subset B_{g_j(t)}(p, 2R)$$

for any point  $p \in D_0$  because  $g_j(t)$  is  $U(n)$ -invariant. By definition, for all  $t \in [a, b]$ ,  $B_{g_j(t)}(p, R)$  subconverges to  $B_{g_\infty(t)}(p_\infty, R)$  in Cheeger-Gromov-Hamilton sense and so  $B_{g_\infty(t)}(p_\infty, R)$  is homeomorphic to  $B_{g_j(t)}(p, R)$  for sufficiently large  $j$ . We then obtain an exhaustion  $B_{g_\infty(t)}(p, R_k)$  with each  $R_k$  sufficiently large and  $R_k \rightarrow \infty$ . Each of them is homeomorphic to  $\mathbb{C}^n$  blown-up at one point. Therefore  $X_\infty$  is homeomorphic to  $\mathbb{C}^n$  blown-up at one point.  $\square$

We remark that the convergence in the above proof is  $U(n)$ -equivariant and the limiting shrinking soliton  $(X_\infty, g_\infty, p_\infty)$  is invariant under a free action of the unitary group  $U(n)$ . We also remark that the Type I blow-up limit is a trivial shrinking soliton if one chooses a fixed base point outside the exceptional divisor  $D_0$ . This is because the flow converges in local  $C^\infty$  topology outside  $D_0$  to a smooth Kähler metric on  $X \setminus D_0$  by Theorem 3.1 [22].

Combining Theorem 4.2 and Proposition 5.1, we complete the proof of Theorem 1.1.

## 6. Some generalizations

In this section, we discuss some generalizations of Theorem 1.1. First, Theorem 1.1 can be easily generalized to  $X_{n,k}$  defined in section 1 by the same argument in the previous sections.

**Theorem 6.1.** *The Ricci flow on  $X_{n,k}$  must develop Type I singularities for any  $G_{n,k}$ -invariant initial Kähler metric.*

Let  $g(t)$  be the smooth solution defined on  $t \in [0, T)$ , where  $T \in (0, \infty)$  is the singular time. For every  $K_j \rightarrow \infty$ , we consider the rescaled Ricci flow  $(X, g_j(t'))$  defined on  $[-K_j T, 0)$  by

$$g_j(t') = K_j g(T + K_j^{-1} t').$$

Then one and only one of the following must occur.

- (1) If  $\liminf_{t \rightarrow T} (T - t)^{-1} \text{Vol}(g(t)) = \infty$ , then  $(X, g_j(t'), p)$  subconverges in Cheeger-Gromov-Hamilton sense to a complete nontrivial shrinking gradient Kähler-Ricci soliton on a complete Kähler manifold homeomorphic to the total space of  $L^{-k} = \mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$ , for any  $p$  in the exceptional divisor.
- (2) If  $\liminf_{t \rightarrow T} (T - t)^{-1} \text{Vol}(g(t)) \in (0, \infty)$ , then  $(X, g_j(t'), p_j)$  subconverges in Cheeger-Gromov-Hamilton sense to  $(\mathbb{C}^{n-1} \times \mathbb{CP}^1, g_{\mathbb{C}^{n-1}} \oplus (-t') g_{FS})$ , where  $g_{\mathbb{C}^{n-1}}$  is the standard flat metric on  $\mathbb{C}^{n-1}$  and  $g_{FS}$  the Fubini-Study metric on  $\mathbb{CP}^1$  for any sequence of points  $p_j$  [7].
- (3) If  $\liminf_{t \rightarrow T} (T - t)^{-1} \text{Vol}(g(t)) = 0$ , then  $(X, g_j(t'))$  converges in Cheeger-Gromov-Hamilton sense to the unique compact shrinking Kähler-Ricci soliton on  $X_{n,k}$  blown-up at one point [24].

We can also consider the Calabi symmetry introduced by Calabi [2] for projective bundles over a Kähler-Einstein manifold (also see [12, 20]). In particular, we can consider the Ricci flow on generalizations of  $X_{n,k}$

$$X_{m,n,k} = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^n} \oplus \mathcal{O}_{\mathbb{CP}^n}(-k)^{\oplus(m+1)}), \quad k = 1, 2, \dots$$

Similar results are obtained for  $X_{m,n,k}$  in [20] for global Gromov-Hausdorff convergence at the singular time, as those for  $X_{n,k}$  in [18]. Furthermore, one can obtain the same lower bound for the holomorphic bisectional curvature as in Proposition 3.1.

**Proposition 6.1.** *Let  $g(t)$  be the solution of the Ricci flow on  $X_{m,n,k}$  for an initial Kähler metric with Calabi symmetry. Then if  $1 \leq m \leq n$  and if*

$$\liminf_{t \rightarrow T} \text{Vol}(g(t)) > 0$$

*where  $T > 0$  is the singular time, then the holomorphic bisectional curvature of  $g(t)$  is uniformly bounded below by  $-\frac{C}{T-t}$  for some constant  $C > 0$ .*

Although we are unable to exclude Type II singularities, one can show by the same argument in section 4, that the universal cover of the blow-up limit is an eternal solution of the Ricci flow which splits into  $\mathbb{C}^n \times N^{m+1}$  flat in  $\mathbb{C}^n$  and  $N^{m+1}$  of nonnegative holomorphic bisectional curvature, if the flow develops Type II singularities. Of course, a Type I bound for the scalar curvature suffices to prove a similar theorem as Theorem 1.1.

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## REFERENCES

- [1] Aubin, T. *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95
- [2] Calabi, E. *Extremal Kähler metrics*, in Seminar on Differential Geometry, pp. 259–290, Ann. of Math. Stud., **102**, Princeton Univ. Press, Princeton, N.J., 1982
- [3] Cao, H.-D. *Existence of gradient Kähler-Ricci solitons*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), 1–16, A K Peters, Wellesley, MA, 1996
- [4] Cao, H.-D. *dimension reduction in the Kähler-Ricci flow*, Comm. Anal. Geom. 12 (2004), no. 1-2, 305–320
- [5] Chow, B. *The Ricci flow on the 2-sphere*, J. Differential Geom. **33** (1991) 325–334
- [6] Feldman, M., Ilmanen, T. and Knopf, D. *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Differential Geom. **65** (2003), no. 2, 169–209
- [7] Fong, T. *On the collapsing rate of the Kähler-Ricci flow with finite-time singularity*, arXiv:1112.5987
- [8] Koiso, N. *On rotationally symmetric Hamilton’s equation for Kähler-Einstein metrics*, Recent topics in differential and analytic geometry, 327–337, Adv. Stud. Pure Math., 18-I, Academic Press, Boston, MA, 1990
- [9] Hamilton, R. S. *Three-manifolds with positive Ricci curvature*, J. Differ. Geom. 17 (1982), no. 2, 255–306
- [10] Hamilton, R. S. *Four-manifolds with positive curvature operator*, J. Differ. Geom. 24 (1986), 153–179

- [11] Hamilton, R.S. *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995
- [12] Li, C. *On rotationally symmetric Kähler-Ricci solitons*, arXiv:1004.4049
- [13] Maximo, D. *On the blow-up of four dimensional Ricci flow singularities*, preprint
- [14] Naber, A. *Noncompact shrinking 4-solitons with nonnegative curvature*, J. Reine Angew. Math. **645** (2010), 125–153
- [15] Perelman, G. *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159
- [16] Perelman, G. unpublished work on the Kähler-Ricci flow
- [17] Phong, D.H. and Sturm, J. *On stability and the convergence of the Kähler-Ricci flow*, J. Differential Geom. **72** (2006), no. 1, 149–168
- [18] Song, J. and Weinkove, B. *The Kähler-Ricci flow on Hirzebruch surfaces*, J. Reine Angew. Math. **659** (2011), 141–168
- [19] Song, J. and Weinkove, B. *Contracting exceptional divisors by the Kähler-Ricci flow*, arXiv:1003.0718
- [20] Song, J. and Yuan, Y. *Metric flips with Calabi ansatz*, to appear in G.A.F.A., arXiv:1011.1608
- [21] Tian, G. *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), no. 1, 1–37
- [22] Tian, G. and Zhang, Z. *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B **27** (2006), no. 2, 179–192
- [23] Tsuji, H. *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. **281** (1988), 123–133
- [24] Wang, X.J. and Zhu, X. *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Advances Math. **188** (2004) 87–103
- [25] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411
- [26] Yau, S.-T. *Open problems in geometry*, Proc. Symposia Pure Math. **54** (1993), 1–28 (problem 65)
- [27] Zhang, Z. *On degenerate Monge-Ampère equations over closed Kähler manifolds*, Int. Math. Res. Not. **2006**, Art. ID 63640, 18 pp
- [28] Zhu, X. *Kähler-Ricci flow on a toric manifold with positive first Chern class*, arXiv:math/0703486

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854

E-mail address: jiansong@math.rutgers.edu